Reputation with Multiple Commitment Types: Continuous-Time

Sambuddha Ghosh and Seokjong Ryu

Shanghai University of Finance and Economics

KAIST seminar

Oct. 10, 2018

What is Reputation?

Small uncertainty has a Magnifying effect in dynamic games

Eg. The Chain Store Paradox

- Complete vs. Incomplete information
- Weak monopolist could build a reputation as Strong.

Literature

In Discrete-time:

- Kreps and Wilson (1982) and Milgrom and Roberts (1982)
- Fudenberg and Levine (1989, 1992)

In Continuous-time:

- Faingold and Sannikov (2011)
  - 2 players
  - continuous-time: $t \in [0, \infty)$
  - single commitment type

Our paper

Our paper study reputation with:

- continuous-time
- 2 players: one long-lived and a continuum of short-lived
- one-sided incomplete information
- **multiple commitment types**

Why is this important?

- Slight uncertainty is important for building reputation
- How to rule out all types but one?
  - Multiple vs. Single
- Robustness check
  - Fudenberg and Levine (1989, 1992): enough if there is the Stackelberg type

Contribution

- Multiple types case cannot be reduced to Single type case
- We characterize a PDE for optimal payoff function
  - Find an approximate Markov equilibrium
- We characterize a condition for optimal actions
- We find a stochastic representation of the approximate solution

Model

An infinite-time horizon dynamic game with imperfect monitoring:

- A long-lived large player chooses $a_t \in A$
  - For $r > 0$,
$$\int_0^\infty re^{-rt} g(a_t, \bar{b}_t) dt$$
- A continuum of infinitely lived small players : $i \in [0, 1]$
  - Each i chooses $b_t^i \in B$ with aggregate dist $\bar{b}_t$
  - For $r > 0$
$$\int_0^\infty re^{-rt} h(a_t, b_t^i, \bar{b}_t) dt$$

Model (cont.)

A type space $\{T_0, T_1, ..., T_K\}$ supports small players' prior:

- T_0 is a normal type with belief $\theta_{0,t}$
- For $k \in \{1, ..., K\}$, T_k is a commitment type with belief $\theta_{k,t}$
  - T_k is believed to play a fixed action $a_k^* \in A$ every time

A belief space is defined:

$$\Delta^{K-1} = \Big\{\theta_t = (\theta_{1,t}, ..., \theta_{K,t}) \in \mathbb{R}_+^K \mid \sum_{k=1}^{K} \theta_{k,t} = 1 - \theta_{0,t} < 1 \text{ and } \theta_{k,t} > 0 \text{ for every } k \in \{1, ..., K\}\Big\}$$

Model (cont.)

Public signals $\{X_t\}_{t \geq 0}$ follows the diffusion process:

$$dX_t = \mu(a_t, \bar{b}_t)dt + \sigma(\bar{b}_t)dB_t$$

- $\{B_t\}_{t \geq 0}$ is a d-dimensional Brownian motion with $d \geq K$
- $\mu(a_t, \bar{b}_t) \in \mathbb{R}^d$ and $\sigma(\bar{b}_t) \in \mathbb{R}^{d \times d}$

The continuation value of T_0 at time $t \geq 0$:

$$W_t(S) = \mathbb{E}_t \left\{ \int_t^{\infty} r e^{-r(s-t)} g(a_s, \bar{b}_s) ds \mid T_0 \right\}$$

where $S = \{(a_s, \bar{b}_s)\}_{s \geq 0}$ is a strategy profile.

Faingold and Sannikov (2011)

A bounded process $\{W_t\}_{t \geq 0}$ is the process of continuation values of the normal type under a public strategy profile $S = \{(a_s, \bar{b}_s)\}_{s \geq 0}$ iff for some $\beta = \{\beta_t\}_{t \geq 0} \in \mathcal{L}$,

$$dW_t = r\big(W_t - g(a_t, \bar{b}_t)\big)dt + r\beta_t \cdot \big(dX_t - \mu(a_t, \bar{b}_t)dt\big)$$

This β determines the reputation factor Z.

Faingold and Sannikov (2011)

$\{\theta_t\}_{t\geq 0} = \{(\theta_{1,t}, ..., \theta_{K,t})\}_{t\geq 0}$ is consistent with $(a_t, \bar{b}_t)_{t\geq 0}$ iff

(a) $(\theta_{1,0}, ..., \theta_{K,0}) = p$ for any given prior $p \in \Delta^{K-1}$

(b) for each $k \in \{0, 1, ..., K\}$ and $t \in [0, \infty)$

$$d\theta_{k,t} = \gamma_k(a_t, \bar{b}_t, \theta_t) \cdot \sigma^{-1}(\bar{b}_t)\big(dX_t - \mu^\theta(a_t, \bar{b}_t)dt\big)$$

- $\gamma_0(a_t, \bar{b}_t, \theta_t) \equiv \theta_{0,t}\sigma^{-1}(\bar{b}_t)\big(\mu(a_t, \bar{b}_t) - \mu^\theta(a_t, \bar{b}_t)\big)$
- $\gamma_k(a_t, \bar{b}_t, \theta_t) \equiv \theta_{k,t}\sigma^{-1}(\bar{b}_t)\big(\mu(a_k^*, \bar{b}_t) - \mu^\theta(a_t, \bar{b}_t)\big)$
- $\mu^\theta(a_t, \bar{b}_t) \equiv \theta_{0,t}\mu(a_t, \bar{b}_t) + \sum_{k=1}^{K} \theta_{k,t}\mu(a_k^*, \bar{b}_t)$

PDE

By letting $W_t = U(\theta_t)$, on $\overline{\Delta^{K-1}} := \Delta^{K-1} \cup \partial\Delta^{K-1}$,

$$\frac{1}{2}\sum_{i,j=1}^{K}\gamma_i\gamma_j U_{\theta_i\theta_j}(\theta) + \sum_{i=1}^{K}\frac{\gamma_0\gamma_i}{\theta_0}U_{\theta_i}(\theta) - rU(\theta) = -rg, \qquad (1)$$

- $a := \mathcal{N}_a\big(\theta, Z\big) \in A \subset \mathbb{R}$
- $b := \mathcal{N}_b\big(\theta, Z\big) \in B \subset \mathbb{R}$
- $Z(\theta) = Z(\theta, \nabla U(\theta))$ is the reputation factor

This is a 2^{nd}-order quasi-linear PDE

Problem 1

The first problem is the Nonlinearity.

- γ_i is a function of U through the optimal actions

We get around this difficulty by an iterative procedure:

1. Pick arbitrary Lipschitz continuous best response functions and transform the quasilinear PDE into an linear elliptic PDE.
2. Take the solution of the linear elliptic PDE and use it to derive the new optimal actions.
3. Return to Step 1 and iterate.

Problem 2

The second problem is Boundary Conditions

- Need the value attained by U we are solving for on $\partial\Delta^{K-1}$
- The boundary conditions in FS(2011) are much simpler

This problem is resolved by using state-of-the-art techniques in the PDE literature:

1. Omit the conditions for those points that are never touched
2. Our PDE would never touch the boundary
3. Every such PDE possesses a continuous solution

Problem 3

The third problem is Iteration

- Need to ensure that the intermediate U_n at each step is differentiable
- Unfortunately, we know that it is only continuous

This is where we use the technique of 'mollification'

1. Use Tietze's extension theorem to extend U_n to $\mathbb{R}^K$
2. Apply mollification to this function on the larger domain
3. Obtain a differentiable function that is defined on a set including $\overline{\Delta^{K-1}}$

As $n \to \infty$

We find a convergent subsequence of U_n:

- Apply the *Arzela-Ascoli Theorem*
- This is a *uniformly* convergent sequence

However, this is not enough:

- Need to show the limit U_n is the solution of the limit PDE
- Apply a version of Trotter-Kato Theorem

As the mollification goes to 0

We show the iterative procedure converges as the mollification goes to zero.

- Apply the *Arzela-Ascoli Theorem*

Finally, we find the common limit of U_n as both $n \to \infty$ and mollification goes to 0

- This is one convergent sequence: not guarantee uniqueness
- Note that this is an *approximate* solution to the original PDE.

Iteration

1. Pick any continuous functions $a_0(\theta)$ and $b_0(\theta)$ on $\overline{\Delta^{K-1}}$.
2. Plug them into $\gamma_i(a_0(\theta), b_0(\theta), \theta) = \gamma_i^0(\theta)$ and $g(a_0(\theta), b_0(\theta)) = g^0(\theta)$.
3. Solve the linear PDE on $\overline{\Delta^{K-1}}$ to find $U_0(\theta)$ that is continuous on $\overline{\Delta^{K-1}}$.
4. Extend $U_0(\theta)$ to a continuous function $U_0(\theta)$ on $\mathbb{R}^K$.
5. Fix $\rho > 0$ and an open subset $\Delta_\rho \subset \mathbb{R}^K$ such that $\overline{\Delta^{K-1}} \subset \Delta_\rho$ and $dist(y, \overline{\Delta^{K-1}}) \leq \rho$ for $y \in \Delta_\rho / \Delta^{K-1}$.
6. Mollify $U_0(\theta)$ on Δ_ρ for $0 < \varepsilon(\rho) < \rho$. Let the mollified function $U_0^{\varepsilon(\rho)}(\theta) \in C^\infty(\Delta_{\varepsilon(\rho)}^{K-1})$ where $\overline{\Delta^{K-1}} \subset \Delta_{\varepsilon(\rho)}^{K-1} \subset \Delta_\rho$.

Iteration (cont.)

7. Let $a_1^{\varepsilon(\rho)}(\theta) = \mathcal{N}_a(\theta, DU_0^{\varepsilon(\rho)}(\theta))$ and $b_1^{\varepsilon(\rho)}(\theta) = \mathcal{N}_b(\theta, DU_0^{\varepsilon(\rho)}(\theta))$.
8. Plug them into $\gamma_i(a_1^{\varepsilon(\rho)}(\theta), b_1^{\varepsilon(\rho)}(\theta), \theta) = \gamma_{i,\varepsilon(\rho)}^1(\theta)$ and $g(a_1^{\varepsilon(\rho)}(\theta), b_1^{\varepsilon(\rho)}(\theta)) = g_{\varepsilon(\rho)}^1(\theta)$.
9. Go to Step 3

$$\vdots$$

$$\vdots$$

$$\vdots$$

Assumptions

(0) The best-response function of the reputation-builder is single valued.

(1) Both best response functions $\mathcal{N}_a(\theta, Z)$ and $\mathcal{N}_b(\theta, Z)$ are Lipschitz continuous in the belief-reputation pair (θ, Z).

(2) For certain $f(\theta)$ belonging to the class of smooth functions on a compact set, the function $Z(\theta) := Z(\theta, f(\theta))$ is Lipschitz continuous in θ:

$$|Z(\theta_1) - Z(\theta_2)| \leq C_{\theta_1,\theta_2}|\theta_1 - \theta_2|$$

where C_{θ_1,θ_2} depends on only θ_1 and θ_2.

Assumptions (cont.)

(3) For every $\theta \in \overline{\Delta^{K-1}}$, $\left\{\mu(a(\theta), b(\theta)), \mu(a_1^*, b(\theta)), ..., \mu(a_K^*, b(\theta))\right\}$ is linearly independent.

(4) For any $\theta = (\theta_1, ..., \theta_K) \in \Delta^{K-1}$, and $a(\theta)$ and $b(\theta)$ on Δ^{K-1},

$$\sum_{k=1}^{K} \left\{ \frac{\theta_k^2}{\sum_{k=1}^{K} \theta_k^2} - \theta_k \right\} \mu(a_k^*, b(\theta)) \neq \theta_0 \mu(a(\theta), b(\theta))$$

(5) The payoff function $g(a, b)$ is uniformly bounded and Lipschitz continuous with $\underline{g} \leq g \leq \bar{g}$.

Non-attainable Boundary

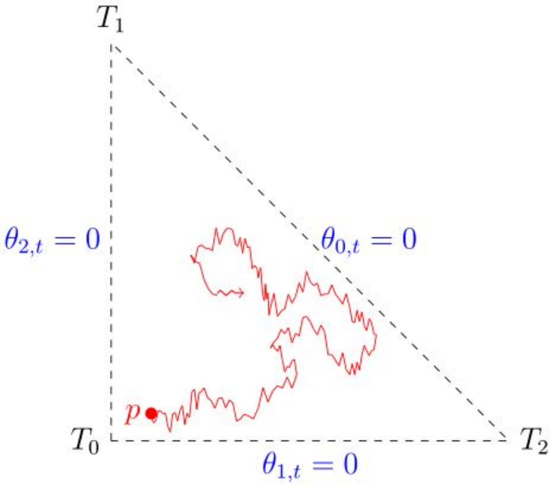
Proposition 1. Under Condition (4), for any $\theta \in \Delta^{K-1}$,

$$P_\theta\left\{\theta_t \in \partial\Delta^{K-1} \text{ for some } t > 0\right\} = 0$$

at each step of 2^{nd}-order linear elliptic PDE.

- Imperfect monitoring
- Every commitment type is alive on the small player's support
- A Multiple types problem cannot be reduced to a Single type problem

When K = 2



Eq. Action Correspondence

Definition. Let $\mathcal{N} : \Delta^{K-1} \times \mathbb{R} \rightrightarrows A \times \Delta(B)$ be a best response correspondence defined by:

$$\mathcal{N}(\theta_t, \mathrm{Z}) \equiv \Big\{ (\mathcal{N}_a(\theta_t, \mathrm{Z}), \mathcal{N}_b(\theta_t, \mathrm{Z})) = (a, b) :$$

$$a \in \mathrm{argmax}_{a' \in A} g(a', \bar{b}) + \left(\sigma(\bar{b})\sigma(\bar{b})^T\right)^{-1} \cdot \mathrm{Z}_t^T \cdot L\big(a, a', \{\mu_k\}_{k=1}^K, \bar{b}\big)$$

$$b \in \mathrm{argmax}_{b' \in B} \; \theta_{0,t} h(a, b', \bar{b}) + \sum_{i=1}^{K} \theta_{i,t} h(a_i^*, b', \bar{b}) \;\; \forall b \in \mathrm{supp}\, \bar{b} \Big\}$$

- a is a *dynamic* optimal action
- b is a *static* optimal action

Eq. Action Correspondence (cont.)

where

- $Z_t^T \equiv -\frac{1}{r}\big(\theta_{1,t}U_{\theta_{1,t}}, ..., \theta_{K,t}U_{\theta_{K,t}}\big) \cdot M(\theta)$

  with $M(\theta) = \begin{pmatrix} (1-\theta_{1,t}) & \theta_{2,t} & \cdots & \theta_{K,t} \\ \theta_{1,t} & (1-\theta_{2,t}) & \cdots & \theta_{K,t} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{1,t} & \theta_{2,t} & \cdots & (1-\theta_{K,t}) \end{pmatrix}$

- $L\big(a, a', \{\mu_k\}_{k=1}^K, \bar{b}\big) = \begin{pmatrix} (\mu_1 - \mu(a,\bar{b})) \cdot \mu(a',\bar{b}) \\ \vdots \\ (\mu_K - \mu(a,\bar{b})) \cdot \mu(a',\bar{b}) \end{pmatrix}$

Existence of Approximate Solution

Theorem 1. Under Assumptions, for any $r > 0$, there exists an approximate Markov equilibrium payoff function $U(\cdot)$ on $\overline{\Delta}^{K-1}$ to the PDE (1):

$$\frac{1}{2}\sum_{i,j=1}^{K} \gamma_i\gamma_j U_{\theta_i\theta_j}(\theta) + \sum_{i=1}^{K} \frac{\gamma_0\gamma_i}{\theta_0} U_{\theta_i}(\theta) - rU(\theta) = -rg,$$

- $a(\theta) := \mathcal{N}_a\big(\theta, Z(\theta)\big) \in A \subset \mathbb{R}$
- $b(\theta) := \mathcal{N}_b\big(\theta, Z(\theta)\big) \in B \subset \mathbb{R}$

where $U(\cdot) = \lim_{\rho\to 0} \lim_{n\to\infty} U_{n,\varepsilon(\rho)}(\cdot)$ for the solution, $U_{n,\varepsilon(\rho)}(\cdot)$, at the n^{th} step PDE with $\varepsilon(\rho)$-mollification.

Equilibrium Degeneracy

Let $\theta_0^* \equiv (0, ..., 0) \in \Delta^{K-1}$ and $\theta_k^* \equiv (0, ..., 0, 1, 0, ..., 0) \in \Delta^{K-1}$ for any $k \in \{1, ..., K\}$ that T_k is a Stackelberg type.

Propositioin 2 Suppose that small players are certain that the large player is either a normal type T_0 or a Stackelberg type T_j for some $j \in \{1, ..., K\}$. Then, for any $r > 0$,

$$U(\theta_j^*) \in g\big(\mathcal{N}(\theta_j^*, r)\big)$$

where U is the approximate Markov equilibrium payoff.

Construction of Restricted Belief Spaces

For sufficiently small $\delta > 0$, suppose that $D^\delta \subset \Delta^{K-1}$ is a convex and connected open subset with boundary ∂D^δ such that

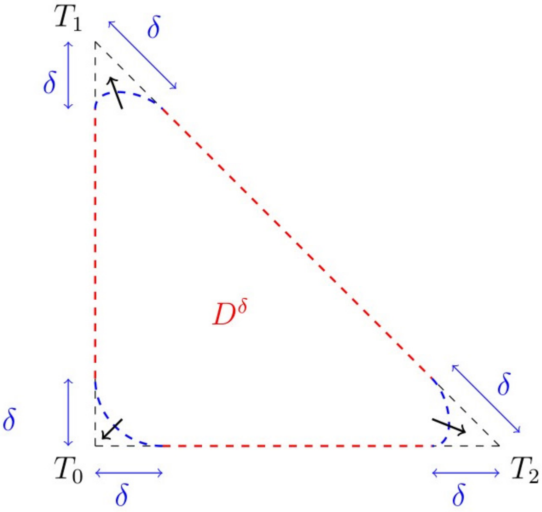
(a) $\cup_{\delta>0} D^\delta = \Delta^{K-1}$ and $D^{\delta_1} \subset D^{\delta_2}$ for any $\delta_1 > \delta_2$

(b) $\partial D^\delta\big|_{\Omega_\delta} = \partial \Delta^{K-1}\big|_{\Omega_\delta}$

(c) $\partial \Delta^\epsilon_\delta \equiv \partial D^\delta \backslash \Omega_\delta \subset \cup_{k\in\{0,1,..,K\}}\{\theta_k > 1-\delta\}$

where Ω_δ is the subset of $\partial \overline{\Delta^{K-1}}$ such that $\delta < \theta_k < 1-\delta$ for some $k \in \{0, 1, ..., K\}$ and $\theta_l = 0$ for some $l \neq k$.

A restricted belief space when K = 2



A reduced PDE

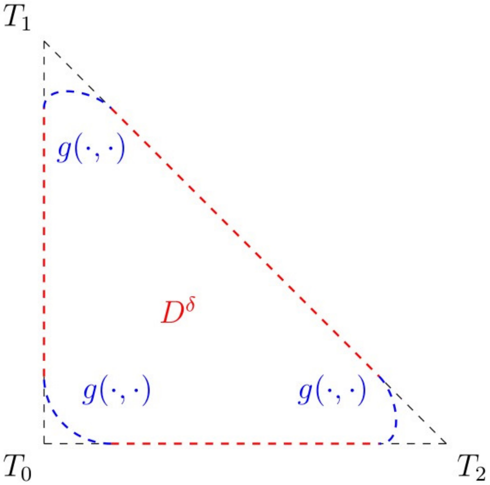
For each $\delta > 0$, $r > 0$, and $\rho > 0$, consider the reduced limit $(\text{as } n \to \infty)$ problem on D^{δ}:

$$\frac{1}{2}\sum_{i,j=1}^{K}\gamma_i^{\rho}\gamma_j^{\rho}U_{\theta_i\theta_j}(\theta) + \sum_{i=1}^{K}\frac{\gamma_0^{\rho}\gamma_i^{\rho}}{\theta_0}U_{\theta_i}(\theta) - rU(\theta) = -rg^{\rho}, \tag{2}$$

$$\begin{aligned}
U^{\delta}(\theta) =& g^{\rho}\big(\mathcal{N}(\theta_0^*, r)\big) \text{ on } \partial\Delta_{\delta}^{\epsilon} \cap \{\theta_0 > 1-\delta\} \\
U^{\delta}(\theta) =& g^{\rho}\big(\mathcal{N}(\theta_1^*, r)\big) \text{ on } \partial\Delta_{\delta}^{\epsilon} \cap \{\theta_1 > 1-\delta\} \\
&\vdots \\
U^{\delta}(\theta) =& g^{\rho}\big(\mathcal{N}(\theta_m^*, r)\big) \text{ on } \partial\Delta_{\delta}^{\epsilon} \cap \{\theta_m > 1-\delta\}
\end{aligned}$$

where $\{1, ..., m\} \subset \{1, ..., K\}$ is a set of Stackelberg types.

A reduced problem when K = 2



Stochastic representation on D^δ

Proposition 3. Under Assumptions, the payoff $U^\delta_{\varepsilon(\rho)}(\theta)$ that is a solution to PDE (2) has the following form on D^δ: for any $\theta \in D^\delta$, $r > 0$, and $\varepsilon(\rho) > 0$,

$$U^\delta_{\varepsilon(\rho)}(\theta) = \mathbb{E}_\theta\left[g(\theta_{\tau^\delta})\exp\{-r\tau^\delta\}\right] + r\mathbb{E}_\theta\int_0^{\tau^\delta} g(\theta_s)\exp\{-rs\}ds$$

where $\tau^\delta \equiv \inf\left\{t > 0 \,\middle|\, \theta_t \notin D^\delta\right\}$. Furthermore, $U^\delta_{\varepsilon(\rho)}(\theta)$ satisfies the followings boundary conditions : for any $j \in \{0, 1, ..., K\}$,

$$U^\delta_{\varepsilon(\rho)}(\theta_{\tau^\delta}) = g^\rho\big(\mathcal{N}(\theta^*_j, r)\big)$$

on $\partial\Delta^\epsilon_\delta \cap \{\theta_j > 1 - \delta\}$

Stochastic representation on Δ^{K-1}

Theorem 2. Under Assumptions, the $\varepsilon(\rho)$-Markov equilibrium payoff $U_{\varepsilon(\rho)}(\theta)$ on Δ^{K-1} is given by: for any given $\theta \in \Delta^{K-1}$ and $r > 0$,

$$U_{\varepsilon(\rho)}(\theta) = \mathbb{E}_\theta\Big[g(\theta_\tau)\exp\{-r\tau\}\Big] + r\mathbb{E}_\theta \int_0^\tau g(\theta_s)\exp\{-rs\}ds$$

where $\tau \equiv \lim_{\delta\to 0} \tau^\delta$.

Furthermore, for each j-th vertex $\theta_j^* \in \partial M$,

$$\lim_{\theta\to\theta_j^*} U_{\varepsilon(\rho)}(\theta) = g^\rho\big(\mathcal{N}(\theta_j^*, r)\big)$$

Conclusion

With multiple commitment types,

- Characterize the optimal equation for the Eq. payoff
- Find an approx. Markov equilibrium
- Find a stochastic representation of the approx. Eq. Payoff

Ongoing research

- What about the convergence of the stochastic representation?
- What about the approximate equilibrium actions?

Future research

- Numerical Analysis?